

Quasi Markovian behavior in mixing maps

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Abstract

We consider the time dependent probability distribution of a coarse grained observable Y whose evolution is governed by a discrete time map. If the map is mixing, the time dependent one-step transition probabilities converge in the long time limit to yield an ergodic stochastic matrix. The stationary distribution of this matrix is identical to the asymptotic distribution of Y under the exact dynamics. The n th time iterate of the baker map is explicitly computed and used to compare the time evolution of the occupation probabilities with those of the approximating Markov chain. The convergence is found to be at least exponentially fast for all rectangular partitions with Lebesgue measure. In particular, uniform rectangles form a Markov partition for which we find exact agreement.

Key words: master equation; mixing; Markov chain; baker map; chaos

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1 Introduction

The technique of coarse graining has often been used to obtain irreversible behavior from systems which, though they may be reversible, are “sufficiently” chaotic [7,12,15]. The procedure is to describe the time evolution of the system in terms of macroscopic variables whose preimages form the cells of the partition. More abstractly, coarse graining may be described as a contraction of the exact dynamics onto a corresponding symbolic dynamics [1]. One of the goals of this approach is be able to describe the macroscopic dynamics in a self-contained form, in other words, to obtain macroscopic laws which do not refer to the underlying microstates. This may be possible if the macroscopic dynamics satisfy a master equation [13].

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It has long been realized, however, that the evolution of observables under reversible microdynamics are generally non-Markovian due to memory effects [12]. While it may be possible to construct a partition for which the resulting symbolic dynamics are exactly Markovian [15,11], such partitions are exceptional and in general do not correspond to the macroscopic variables one is interested in. Assumptions of molecular chaos have in the past been used to argue that the Markov assumption is well approximated for complex macroscopic systems [12]; however, it is unclear whether such assumptions would be consistent with the underlying microscopic dynamics. What are needed are sufficient conditions on the dynamics, as well as the macroscopic observables, to ensure that the Markov assumption is well approximated. This has been done, e.g., for quantum systems weakly interacting with a heat bath [5], but it is not clear how those results may apply in more general settings. Since the master equation is characterized by only short-term memory effects, it is natural to suppose that decaying correlations should at least be a necessary condition. Thus, one is led to consider systems which are mixing.

We begin in Section 2 with a discussion of the general problem for discrete time maps. An equation for the time dependent occupation probabilities, i.e. the generalized master equation, is derived. (Since it involves one-step transition probabilities that are in general dependent upon both the initial cell occupied and the iteration time, the generalized master equation is not generally a true master equation.) In Section 3 we consider the asymptotic behavior of these transition probabilities for long times. For mixing maps it is shown that the one-step transition probabilities converge to an ergodic stochastic matrix. The corresponding Markov chain is found to converge to the same stationary distribution as that of the exact process. The baker map is considered in Section 4 as an exactly solvable system which nevertheless contains many of the important features of more physical systems. An explicit expression is obtained for the n th iterate of a point under this map, thus allowing us to compute exact time dependent occupation probabilities. For rectangular cells with Lebesgue measure it is shown that these probabilities converge at least exponentially fast. In Section 5 we compare the occupation and transition probabilities determined by the baker map with their corresponding approximations using the approximating Markov chain. In particular it is shown that for the baker map any uniform rectangular partition is a Markov partition for which the Markov approximation is exact. The main conclusions are summarized and discussed in Section 6.

2 Generalized Master Equation

Let (X, \mathcal{A}, P) be a probability space and let ϕ be a measurable map on X . We assume that P has a density ρ with respect to some finite measure μ , i.e.

$P \prec \mu$, and that ϕ is μ -measure preserving, i.e. $\mu \circ \phi^{-1} = \mu$. For $x_0 \in X$ and $n \in \mathbb{Z}$, $\phi^n(x_0)$ is the n^{th} iterate of x_0 under ϕ . (If ϕ is invertible, then $\phi^n = (\phi^{-1})^{|n|}$ for $n < 0$; otherwise, ϕ^n is undefined.) The preimage of a measurable set A under ϕ^n will be denoted by $\phi^{-n}(A)$ and represents the set of points that map into A after n time steps. Thus, $P[\phi^{-n}(A)]$ is the probability that an iterate will be in A at time n .

We wish to consider the evolution of a phase function Y of the form

$$Y(x) \equiv \sum_{i \in I} y_i \mathbf{1}_{C_i}(x) , \quad (1)$$

where I is a finite index set, $\{y_i\}_{i \in I}$ are distinct real numbers, and $\mathbf{1}_{C_i}$ is the indicator (characteristic) function on $C_i \in \mathcal{A}$. Without loss of generality, we may assume $\{C_i\}_{i \in I}$ is a disjoint partition of X . We further suppose that the partition is nontrivial in the sense that $\mu[C_i] > 0$ for all $i \in I$. Let $Y_n \equiv Y \circ \phi^n$ and define the transition probabilities $T_{i,j}^{(n)}$ and $T_{i,k,j}^{(m,n)}$ as follows:

$$T_{i,j}^{(n)} \equiv P[\phi^{-n}(C_j) \mid C_i] , \quad (2)$$

$$T_{i,k,j}^{(m,n)} \equiv P[\phi^{-n}(C_j) \mid \phi^{-m}(C_k) \cap C_i] , \quad (3)$$

for $P[C_i] > 0$ and $P[\phi^{-m}(C_k) \cap C_i] > 0$, respectively, and zero otherwise. We see that $T_{i,j}^{(n)}$ is the probability that $Y_n = y_j$ given that $Y_0 = y_i$, and $T_{i,k,j}^{(m,n)}$ is the probability that $Y_n = y_j$ given that $Y_m = y_k$ and $Y_0 = y_i$. In general, $T_{i,k,j}^{(m,n)}$ will depend upon i , so that $\{Y_n\}_{n \in \mathbb{Z}}$ is generally a non-Markovian random process.

Using Eqs. (2) and (3) we obtain the generalized Chapman-Kolmogorov equation for this process:

$$T_{ij}^{(n+m)} = \sum_k T_{ik}^{(n)} T_{ikj}^{(n,n+m)} . \quad (4)$$

Let $W_{ikj}^{(n)} \equiv T_{ikj}^{(n,n+1)}$ denote the one-step transition probability at time n conditioned on C_i . We see that $W_{ikj}^{(n)}$ is the probability that an iterate in cell C_k at time n will make a transition to cell C_j at the next time step, given that it started in cell C_i . Using Eq. (4) it can be shown by induction over n that

$$T_{ij}^{(n)} = \sum_{k_1} \cdots \sum_{k_{n-1}} \prod_{m=1}^n W_{i,k_{m-1},k_m}^{(m-1)} , \quad (5)$$

where $k_0 = i$ and $k_n = j$. Let $p_j(n) \equiv P[\phi^{-n}(C_j)]$ denote the occupation

probability of cell C_j at time n and note that

$$p_j(n) = \sum_i p_i(0) T_{ij}^{(n)} . \quad (6)$$

This is the generalized master equation for the exact occupation probabilities. In general, $W_{ikj}^{(n)}$ depends on both i and n , so Eq. (6) contains memory effects, i.e. the process is not Markovian. Nevertheless we might expect quasi Markovian behavior for maps with decaying correlations. This suggests that mixing may be a key property to consider.

3 Mixing and Approximating Markov Chains

The map ϕ defined in Sect. 2 will be *mixing* with respect to μ [10] if for all measurable sets A and B ,

$$\lim_{n \rightarrow \infty} \mu[\phi^{-n}(A) \cap B] = P_*[A] \mu[B] , \quad (7)$$

where $P_* \equiv \frac{1}{\mu[X]} \mu$. From Eq. (7) it follows [10, p. 72] that if $P \prec \mu$ and $P[B] > 0$ then

$$\lim_{n \rightarrow \infty} P[\phi^{-n}(A) \mid B] = P_*[A]. \quad (8)$$

Using Eq. (8) we find that for $P[C_i] > 0$,

$$\lim_{n \rightarrow \infty} T_{i,j}^{(n)} = \lim_{n \rightarrow \infty} P[\phi^{-n}(C_j) \mid C_i] = P_*[C_j] , \quad (9)$$

the limit being zero otherwise. Note that from Eqs. (6) and (9) it follows that $p_j(n) \rightarrow P_*[C_j]$ as $n \rightarrow \infty$. Thus, mixing implies an asymptotic convergence of solutions to Eq. (6) for any initial $P \prec \mu$. From Eq. (8) we also note that if $P[C_i] > 0$ then $P[\phi^{-n}(C_j) \cap C_i] > 0$ for all n sufficiently large. (Recall that $P_*[C_j] > 0$ for a nontrivial partition.) Thus for $P[C_i] > 0$ we similarly have

$$\lim_{n \rightarrow \infty} T_{i,k,j}^{(n,n+m)} = \lim_{n \rightarrow \infty} \frac{P[\phi^{-n}(\phi^{-m}(C_j) \cap C_k) \mid C_i]}{P[\phi^{-n}(C_k) \mid C_i]} = P_*[\phi^{-m}(C_j) \mid C_k] .$$

The one-step transition probabilities therefore have the following asymptotic form, independent of k :

$$\lim_{n \rightarrow \infty} W_{k,i,j}^{(n)} = P_*[\phi^{-1}(C_j) \mid C_i] \equiv W_{i,j} . \quad (10)$$

Since W is clearly a stochastic matrix, it must correspond to some Markov chain $\{\tilde{Y}_n\}_{n \in \mathbb{Z}}$ on the state space $\{y_i\}_{i \in I}$. We shall refer to $\{\tilde{Y}_n\}_{n \in \mathbb{Z}}$ as the *approximating Markov chain* for the exact process $\{Y_n\}_{n \in \mathbb{Z}}$. Let $q_i(n)$ denote the probability that $\tilde{Y}_n = y_i$ for $i \in I$. In contrast to Eq. (6), we have simply [13, p. 95]

$$q_j(n) = \sum_i q_i(0)(W^n)_{i,j} . \quad (11)$$

The condition $T_{i,j}^{(n)} = (W^n)_{i,j}$ is thus seen to be the necessary and sufficient condition for exact agreement between $\{Y_n\}$ and $\{\tilde{Y}_n\}$, assuming $p_j(0) = q_j(0)$ [11]. The following theorem establishes that, even without this equality, the two processes converge to the same stationary distribution.

Theorem 1 *If ϕ is mixing with respect to a finite invariant measure μ and $\{C_i\}_{i \in I}$ is a nontrivial finite partition of X , then any approximating Markov chain is ergodic.*

PROOF. A Markov chain is ergodic if it is irreducible, aperiodic, and possesses a stationary distribution [16, p. 109]. Let $i, j \in I$ be fixed and take $P = P_*$ in Eq. (5). Given that ϕ is mixing, we know $T_{i,j}^{(n)} \rightarrow P_*[C_j]$. Since $\mu[C_j] > 0$ by assumption, $P_*[C_j] > 0$ and so $T_{i,j}^{(n)} > 0$ for all $n > N_{i,j}$, where $N_{i,j}$ is some positive integer. Thus,

$$\begin{aligned} 0 < T_{i,j}^{(n)} &= \sum_{k_1, \dots, k_{n-1} \in K_n} \prod_{m=1}^n P_*[\phi^{-m}(C_{k_m}) \mid \phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] \\ &\leq \frac{1}{p_n} \sum_{k_1, \dots, k_{n-1} \in K_n} \prod_{m=1}^n P_*[\phi^{-m}(C_{k_m}) \cap \phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] , \end{aligned}$$

where $k_1 = i$, $k_n = j$, and

$$K_n = \left\{ k_1, \dots, k_{n-1} \in I : \prod_{m=1}^n P_*[\phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] > 0 \right\} , \quad (12)$$

$$p_n = \min_{k_1, \dots, k_{n-1} \in K_n} \left\{ \prod_{m=1}^n P_*[\phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] \right\} . \quad (13)$$

We are assured that K_n is nonempty since, were it empty, $T_{i,j}^{(n)}$ would be zero. Since we have taken n to be sufficiently large so that $T_{i,j}^{(n)} > 0$, it follows that K_n is indeed nonempty, and so $p_n > 0$. Multiplying both sides of the inequality

by p_n we conclude that

$$\sum_{k_1, \dots, k_{n-1} \in K_n} \prod_{m=1}^n P_*[\phi^{-m}(C_{k_m}) \cap \phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] > 0 .$$

Now, since $W_{i,j} \geq P_*[\phi^{-1}(C_j) \cap C_i]$ and P_* is invariant,

$$\begin{aligned} (W^n)_{i,j} &= \sum_{k_1} \cdots \sum_{k_{n-1}} \prod_{m=1}^n W_{k_{m-1}, k_m} \\ &\geq \sum_{k_1} \cdots \sum_{k_{n-1}} \prod_{m=1}^n P_*[\phi^{-1}(C_{k_m}) \cap C_{k_{m-1}}] \\ &= \sum_{k_1} \cdots \sum_{k_{n-1}} \prod_{m=1}^n P_*[\phi^{-m}(C_{k_m}) \cap \phi^{-(m-1)}(C_{k_{m-1}})] \\ &\geq \sum_{k_1, \dots, k_{n-1} \in K_n} \prod_{m=1}^n P_*[\phi^{-m}(C_{k_m}) \cap \phi^{-(m-1)}(C_{k_{m-1}}) \cap C_i] . \end{aligned}$$

It then follows that $(W^n)_{i,j} > 0$ for $n > N_{i,j}$. Since i and j were arbitrary, this proves irreducibility.

A Markov chain is aperiodic if it has period $d = 1$. If we suppose that $d > 1$, however, then for $1 \leq r < d$ we would have that $(W^{md+r})_{i,i} = 0$ for all $m \in \mathbb{N}$. We have just shown, however, that $(W^n)_{i,i} > 0$ for all n sufficiently large. Since d cannot be smaller than 1, we conclude that $d = 1$.

Finally, it may be readily verified that $q_i(0) = P_*[C_i]$ for $i \in I$ is a stationary state of W . \square

An ergodic Markov chain will converge to a unique stationary distribution. We thus conclude that for all i, j

$$\lim_{n \rightarrow \infty} (W^n)_{i,j} = P_*[C_j] . \quad (14)$$

The limit is both geometric in rate and uniform in i, j [4, p. 124]. We conclude that both $q_j(n)$ and $p_j(n)$ approach the same asymptotic value of $P_*[C_j]$. The asymptotic distribution will be uniform whenever the partition is uniform, i.e. when $\mu[C_i] = \mu[C_j]$ for all $i, j \in I$, and the following theorem gives the necessary and sufficient conditions for this to be true.

Theorem 2 *The partition is uniform if and only if W is doubly stochastic.*

PROOF. If the partition is uniform then

$$\sum_i W_{i,j} = \frac{1}{\mu[C_1]} \sum_i \mu[\phi^{-1}(C_j) \cap C_i] = \frac{1}{\mu[C_1]} \mu[\phi^{-1}(C_j)] = 1 .$$

Now suppose W is doubly stochastic and note that for any $n \in \mathbb{N}$, W^n will also be doubly stochastic. Since the partition is finite,

$$1 = \lim_{n \rightarrow \infty} \sum_i (W^n)_{i,j} = \sum_i \lim_{n \rightarrow \infty} (W^n)_{i,j} = \sum_i P_*[C_j] .$$

If there are M partition cells, then $\mu[C_j] = \frac{1}{M} \mu[X]$ for all $j \in I$. \square

4 The Baker Map

We now consider the above results in light of a particular example, the baker map, which is defined on $X = [0, 1)^2$ as follows [10]:

$$\phi(x, y) = \left(2x, \frac{y}{2}\right) \mathbf{1}_{[0, 1/2)}(x) + \left(2x - 1, \frac{y}{2} + \frac{1}{2}\right) \mathbf{1}_{[1/2, 1)}(x) . \quad (15)$$

The baker map is a Lebesgue measure preserving map of the Bernoulli type and hence is mixing [2]. It is also time reversal invariant in the sense that $\phi \circ R = R \circ \phi^{-1}$ for the transformation $R(x, y) = (y, x)$. Thus, it possesses all the salient features of a closed Hamiltonian system with time reversal invariance while permitting a simple and exact analysis. An example of an real-world optical system which produces a baker transformation is described in [8].

We wish to explicitly compute the probability $p_j(n)$ defined in Section 2. To do so we will need to calculate probabilities using the given initial probability density. Now, given a probability density ρ , the density after n time steps is given by $U^n \rho$, where U is the Perron-Frobenius operator. Although the generalized spectral decomposition of U (and hence U^n) for the baker map has been constructed [9], it cannot generally be used for computing probabilities. However, since ϕ is invertible and area preserving, $U^n \rho = \rho \circ \phi^{-n}$ [10, p. 47]. Furthermore, since $\phi^{-n} = R \circ \phi^n \circ R$, to compute U^n for $n \in \mathbb{Z}$ it suffices to determine ϕ^n for $n \in \mathbb{N}$.

Theorem 3 *Let ϕ denote the baker map and let U be the corresponding*

Perron-Frobenius operator. For any integer $n \in \mathbb{N}$ and point $(x, y) \in [0, 1]^2$,

$$\phi^n(x, y) = \sum_{m=1}^{2^n} \left(2^n x - m + 1, \frac{y}{2^n} + \alpha(m) \right) \mathbf{1}_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x), \quad (16)$$

where

$$\alpha(m) \equiv \begin{cases} 0, & \text{if } m = 1, \\ \alpha(m - 2^{n-1}) + 1/2^n, & \text{if } m \in \{2^{n-1} + 1, \dots, 2^n\}. \end{cases} \quad (17)$$

PROOF. The proof is by induction on n . For $n = 1$ Eq. (16) clearly holds. By Eq. (15) we have, for $\phi^n(x, y) = \phi(\phi^{n-1}(x, y))$,

$$\begin{aligned} \phi^n(x, y) &= \sum_{m=1}^{2^{n-1}} \left(2^n x - m + 1, \frac{y}{2^n} + \alpha(m) \right) \mathbf{1}_{\left[\frac{m-1}{2^{n-1}}, \frac{m}{2^{n-1}}\right)}(2x) \mathbf{1}_{[0, 1/2)}(x) \\ &\quad + \sum_{m=1}^{2^{n-1}} \left(2^n x - 2^{n-1} - m + 1, \frac{y}{2^n} + \frac{1}{2^n} + \alpha(m) \right) \\ &\quad \times \mathbf{1}_{\left[\frac{m-1}{2^{n-1}}, \frac{m}{2^{n-1}}\right)}(2x - 1) \mathbf{1}_{[1/2, 1)}(x) \end{aligned}$$

In the second sum we make the substitution $m = m' - 2^{n-1}$ and note that $\frac{1}{2^n} + \alpha(m) = \alpha_n(m')$, by Eq. (17). Finally, we note the following identities:

$$\mathbf{1}_{\left[\frac{m-1}{2^{n-1}}, \frac{m}{2^{n-1}}\right)}(2x) \mathbf{1}_{[0, 1/2)}(x) = \mathbf{1}_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x),$$

since $m \leq 2^{n-1}$ is equivalent to $m/2^n \leq \frac{1}{2}$, and

$$\mathbf{1}_{\left[\frac{m-1}{2^{n-1}}, \frac{m}{2^{n-1}}\right)}(2x - 1) \mathbf{1}_{[1/2, 1)}(x) = \mathbf{1}_{\left[\frac{m'-1}{2^n}, \frac{m'}{2^n}\right)}(x),$$

since $m' \geq 2^{n-1} + 1$ is equivalent to $(m' - 1)/2^n \geq \frac{1}{2}$. Thus,

$$\begin{aligned} \phi^n(x, y) &= \sum_{m=1}^{2^{n-1}} \left(2^n x - m + 1, \frac{y}{2^n} + \alpha(m) \right) \mathbf{1}_{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right)}(x) \\ &\quad + \sum_{m'=2^{n-1}+1}^{2^n} \left(2^n x - m' + 1, \frac{y}{2^n} + \alpha(m') \right) \mathbf{1}_{\left[\frac{m'-1}{2^n}, \frac{m'}{2^n}\right)}(x), \end{aligned}$$

which is identical to Eq. (16). \square

Note that the sum in Eq. (16) has only one nonzero term, namely, $m = m_n(x) \equiv \lfloor 2^n x \rfloor + 1$. Thus it may be written more compactly as

$$\phi^n(x, y) = \left(2^n x \bmod 1, \frac{y}{2^n} + \alpha(m_n(x)) \right). \quad (18)$$

Using Eq. (16) and the fact that $U^n \rho = \rho \circ (R \circ \phi^n \circ R)$, we find that for a rectangular region $C = [a, b] \times [c, d]$,

$$P[\phi^{-n}(C)] = \sum_{l=m_n(c)}^{m_n(d)} \int_{\gamma_{n,l}(c)}^{\delta_{n,l}(d)} \int_{\alpha(l)+a/2^n}^{\alpha(l)+b/2^n} \rho(u, v) \, du \, dv, \quad (19)$$

where $\gamma_{n,l}(c) = \max\{2^n c, l-1\} - l + 1$ and $\delta_{n,l}(d) = \min\{2^n d, l\} - l + 1$. (We define $m_n(1) \equiv 2^n$.) Note that $\gamma_{n,l}(c) = 2^n c \bmod 1$ for $l = m_n(c)$ and is zero otherwise, while $\delta_{n,l}(d) = 2^n d \bmod 1$ for $l = m_n(d)$ and is one otherwise.

Let us now consider an initial density of the form

$$\rho(x, y) \equiv \sum_i p_i(0) / \mu_L[C_i] \, \mathbf{1}_{C_i}(x, y), \quad (20)$$

where the $p_i(0)$'s are given constants and $\mu_L[C_i]$ is the Lebesgue measure of cell C_i . (Of course, such a density may be obtained by pre-coarse graining an arbitrary ρ , in which case $p_i(0) = \int_{C_i} \rho(x, y) \, dx \, dy$.) If the cells are rectangular with $C_i = [a_i, b_i] \times [c_i, d_i]$ then

$$T_{i,j}^{(n)} = \mu_L[\phi^{-n}(C_j) \mid C_i] = \sum_{l=m_n(c_j)}^{m_n(d_j)} \frac{f_n[\alpha(l)] \, g_{n,l}}{(b_i - a_i)(d_i - c_i)}, \quad (21)$$

where

$$f_n(\alpha) \equiv \vartheta(\min\{b_i, \alpha + b_j/2^n\} - \max\{a_i, \alpha + a_j/2^n\}), \quad (22)$$

$$g_{n,l} \equiv \vartheta(\min\{d_i, \delta_{n,l}(d_j)\} - \max\{c_i, \gamma_{n,l}(c_j)\}). \quad (23)$$

(The function ϑ is defined such that $\vartheta(x) = x$ for $x > 0$ and $\vartheta(x) = 0$ otherwise.) The occupation probability $p_j(n)$ is now found using Eq. (6).

As an example, consider $\rho = \frac{1}{\mu_L[C_j]} \mathbf{1}_{C_j}$. For $n < \log_2(1/d_j)$ we have

$$p_j(n) = \frac{1}{2^n} \frac{\vartheta(b_j - 2^n a_j) \vartheta(d_j - 2^n c_j)}{(b_j - a_j)(d_j - c_j)} \leq \frac{1}{2^n}. \quad (24)$$

If $a_j = c_j = 0$, we have an initially exponential decay of $p_j(n)$ before it converges to the asymptotic value $\mu_L[C_j] = b_j d_j$.

This example suggests a general convergence rate which is at least exponentially fast. Let us consider the rate of convergence of $T_{i,j}^{(n)}$. Assuming $n > \log_2[(b_j - a_j)/(b_i - a_i)]$ and $n \gg \log_2(d_j - c_j)^{-1}$, we have

$$\mu_L[C_i] T_{i,j}^{(n)} = f_n[\alpha(l_c)] g_{n,l_c} + f_n[\alpha(l_d)] g_{n,l_d} + \sum_{l=l_c+1}^{l_d-1} f_n[\alpha(l)] ,$$

where $l_c = m_n(c_j)$ and $l_d = m_n(d_j)$. The sequence $\alpha(1), \alpha(2), \dots$ is dense in $[0, 1)$, taking values which are uniformly distributed throughout the interval. Thus, we may approximate the sum by an integral for large n .

$$\sum_{l=l_c+1}^{l_d-1} f_n[\alpha(l)] \approx 2^n (d_i - c_i)(d_j - c_j) \int_0^1 f_n(\alpha) d\alpha ,$$

where we note that

$$f_n(\alpha) = \begin{cases} \alpha - a_i + b_j/2^n, & \text{if } a_i - \frac{b_j}{2^n} \leq \alpha < a_i - \frac{a_j}{2^n}, \\ (b_j - a_j)/2^n & \text{if } a_i - \frac{a_j}{2^n} \leq \alpha < b_i - \frac{b_j}{2^n}, \\ b_i - a_j/2^n - \alpha & \text{if } b_i - \frac{b_j}{2^n} \leq \alpha < b_i - \frac{a_j}{2^n}, \\ 0 , & \text{otherwise .} \end{cases} \quad (25)$$

Upon integrating we find that

$$|T_{i,j}^{(n)} - \mu_L[C_j]| \approx \frac{f_n[\alpha(l_c)] g_{n,l_c} + f_n[\alpha(l_d)] g_{n,l_d}}{\mu_L[C_i]} \leq \frac{2(b_j - a_j)}{2^n \mu_L[C_i]} .$$

Thus we have convergence which is at least exponentially fast. (Recall that convergence for the approximating Markov chain is always exponential.)

5 Comparison of Exact and Approximating Processes

We have found that for mixing maps both $p_j(n)$ and $q_j(n)$ converge to the same limit, $P_*[C_j]$, as $n \rightarrow \infty$. If furthermore $p_j(0) = q_j(0)$ for all $j \in I$, then we expect only transient deviations of $p_j(n)$ from its Markovian counterpart

$q_j(n)$. The difference between these two probabilities may be measured by an average over all cells of the ℓ^∞ norms:

$$D_p \equiv \frac{1}{M} \sum_{j=1}^M \sup_{n \in \mathbb{N}} |p_j(n) - q_j(n)|, \quad (26)$$

where M is the total number of cells. Note that $0 \leq D_p \leq 1$, the value of D_p depending upon both the initial measure P and the partition chosen. In particular, if $P = P_*$ then $p_j(n) = P_*[C_j] = q_j(n)$ and $D_p = 0$.

We now return to the baker map and consider a partition of two cells: $C_1 = [0, s) \times [0, 1)$ and $C_2 = [s, 1) \times [0, 1)$. For $s \leq \frac{1}{2}$, the transition matrix given by Eq. (10) is

$$W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{s}{2(1-s)} & \frac{2-3s}{2(1-s)} \end{pmatrix}. \quad (27)$$

The transition matrix for $s > \frac{1}{2}$ is obtained by interchanging s with $1-s$ and taking the transpose.

In Fig. 1 we display the average deviation D_p for $0 < s < 1$. The asymmetry in the graph is due to the fact that the initial measure is on C_1 ; a mirror image of it is obtained if C_2 is used as the starting cell. For s near 0, $\frac{1}{2}$, or 1 we note that D_p goes to zero, implying exact agreement between $p_j(n)$ and $q_j(n)$ for all j and n . This can be understood as a consequence of the following theorem:

Theorem 4 *Let $R_{i,j} = [\frac{i-1}{M}, \frac{i}{M}] \times [\frac{j-1}{N}, \frac{j}{N}]$ for $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$ and $M, N \in \mathbb{N}$. Then $\{R_{i,j}\}$ is a Markov partition for the baker map.*

A proof of this theorem is given in the appendix. Since the initial measure was taken to be uniform with respect to the maximum entropy measure μ_L , it follows that $\{Y_n\}$ is a true Markov chain [3].

6 Discussion

We have shown that for mixing maps the time dependent occupation and one-step transition probabilities, $p_j(n)$ and $W_{kij}^{(n)}$ respectively, converge to values which are independent of both the index k and the initial probability density ρ . The resulting asymptotic transition probabilities W_{ij} form an ergodic

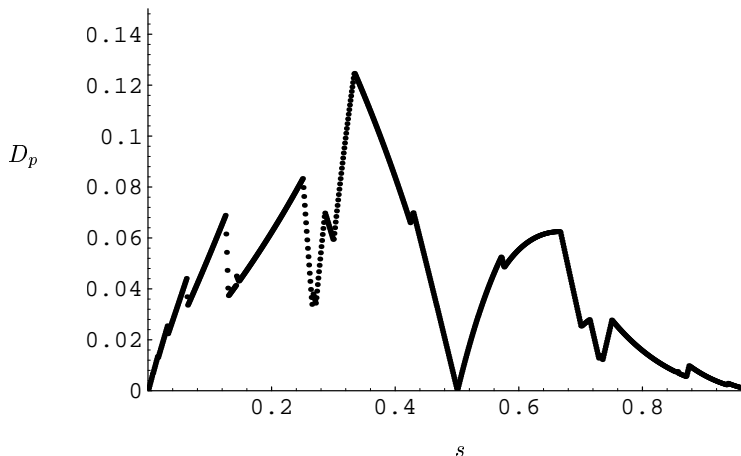


Fig. 1. The above plot shows the values of D_p for a partition of two vertical cells of width s and $1 - s$. Note that D_p goes to zero near $s = 0, \frac{1}{2}, 1$.

stochastic matrix whose unique asymptotic distribution is identical to that of the exact, non-Markovian, process.

To examine the detailed evolution of $p_j(n)$ we considered the baker map ϕ . An explicit expression for $\phi^n(x, y)$ was found with which we were able to determine explicit formulas for $p_j(n)$ in the case of an initial density which is uniform on rectangular cells. It was shown that in this case the occupation probability $p_j(n)$ converges at least exponentially fast, i.e. no slower than $q_j(n)$.

Finally, we studied the average deviation between $p_j(n)$ and $q_j(n)$, which we denoted by D_p . Partitions of two vertical rectangular cells were considered for various cell widths. It was found that agreement was best when all cells were similar, but overall agreement was generally good ($D_p < 0.15$). In particular, uniform rectangular partitions gave exact agreement ($D_p = 0$). This was understood to be a consequence of the fact that such partitions are in fact Markov partitions for the baker map. In general, however, the agreement between the exact and approximating processes will depend upon how strongly the system is mixing. Since the baker map is not only mixing but also a Bernoulli system, it is not clear whether such good agreement is typical of most mixing systems.

While the time evolution of a coarse grained observable under a deterministic map may not be Markovian, for mixing maps this evolution is at least approximately Markovian in the sense that both $\{Y_n\}$ and $\{\tilde{Y}_n\}$ have the same distribution initially and in the distant future. (If the map is invertible then they also have the same distribution in the distant past.) Thus, the two processes do not differ qualitatively and in some cases agree quite well quantitatively.

We have not addressed the problem of obtaining closed form deterministic laws for macroscopic variables. This may be possible for “many to one” observ-

ables corresponding to strongly nonuniform partitions. In general, however, such macroscopic laws are expected to be only approximately deterministic, as befits any realistic model.

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Appendix

Here we prove the claim that every uniform rectangular partition is a Markov partition for the baker map. Note that this is not a true partition since the boundaries of the cells overlap; however, the boundaries are of measure zero and may easily be modified to form a true partition.

PROOF. (Theorem 4) The collection $\{R_{i,j}\}$ of rectangles will be a Markov partition provided $(x, y) \in \text{int } R_{i,j}$ and $\phi(x, y) \in \text{int } R_{i',j'}$ imply that $\phi(W_{R_{i,j}}^u(x, y)) \supseteq W_{R_{i',j'}}^u(\phi(x, y))$ and $\phi(W_{R_{i,j}}^s(x, y)) \subseteq W_{R_{i',j'}}^s(\phi(x, y))$, where $W_{R_{i,j}}^u(x, y)$ and $W_{R_{i,j}}^s(x, y)$ are the intersections with $R_{i,j}$ of the unstable and stable manifolds, respectively, at (x, y) [14]. For the baker map we have $W_{R_{i,j}}^u(x, y) = [\frac{i-1}{M}, \frac{i}{M}] \times \{y\}$ and $W_{R_{i,j}}^s(x, y) = \{x\} \times [\frac{j-1}{N}, \frac{j}{N}]$. We consider the following three cases separately:

Case 1: $\frac{i}{M} \leq \frac{1}{2}$

Since $x < \frac{i}{M} \leq \frac{1}{2}$, $(x', y') = \phi(x, y) = (2x, \frac{y}{2})$ and so

$$\begin{aligned}\phi(W_{R_{i,j}}^u(x, y)) &= [\frac{2i-2}{M}, \frac{2i}{M}] \times \{\frac{y}{2}\} \\ \phi(W_{R_{i,j}}^s(x, y)) &= \{2x\} \times [\frac{j-1}{2N}, \frac{j}{2N}]\end{aligned}$$

and

$$\begin{aligned}W_{R_{i',j'}}^u(\phi(x, y)) &= [\frac{i'-1}{M}, \frac{i'}{M}] \times \{\frac{y}{2}\} \\ W_{R_{i',j'}}^s(\phi(x, y)) &= \{2x\} \times [\frac{j'-1}{N}, \frac{j'}{N}].\end{aligned}$$

We now must determine i', j' in terms of i, j . Note that if $\frac{i-1}{M} < x < \frac{i}{M} - \frac{1}{2M}$ then $\frac{2i-2}{M} < x' < \frac{2i-1}{M}$, so $i' = 2i - 1$. Similarly, if $\frac{i}{M} - \frac{1}{2M} \leq x < \frac{i}{M}$ then

$i' = 2i$. Note also that $\frac{i-1}{N} < y < \frac{j}{N}$ implies $\frac{i-1}{2N} < y' < \frac{j}{2N}$, so $j' = \frac{j}{2}$ if j is even and $j' = \frac{j+1}{2}$ when j is odd. Thus,

$$\begin{aligned} x < \frac{i}{M} - \frac{1}{2M} & : W_{R_{i',j'}}^u(\phi(x, y)) = [\frac{2i-2}{M}, \frac{2i-1}{M}] \times \{\frac{y}{2}\} , \\ \frac{i}{M} - \frac{1}{2M} \leq x & : W_{R_{i',j'}}^u(\phi(x, y)) = [\frac{2i-1}{M}, \frac{2i}{M}] \times \{\frac{y}{2}\} , \\ j \text{ even} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x\} \times [\frac{j-2}{2N}, \frac{j}{2N}] , \\ j \text{ odd} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x\} \times [\frac{j-1}{2N}, \frac{j+1}{2N}] . \end{aligned}$$

Case 2: $\frac{i-1}{M} \leq \frac{1}{2} < \frac{i}{M}$

For this case M is necessarily odd and $i = \frac{M+1}{2}$. We consider the cases $x < \frac{1}{2}$ and $x \geq \frac{1}{2}$ separately. For both cases we have

$$\phi(W_{R_{i,j}}^u(x, y)) = [1 - \frac{1}{M}, 1] \times \{\frac{y}{2}\} \cup [0, \frac{1}{M}] \times \{\frac{y+1}{2}\} .$$

Case 2a: $x < \frac{1}{2}$

Since $x < \frac{1}{2}$, $(x', y') = \phi(x, y) = (2x, \frac{y}{2})$. From the fact that $\frac{M-1}{2M} \leq x < \frac{1}{2}$ it follows that $\frac{M-1}{M} \leq x' < 1$, so $i' = M$ and

$$W_{R_{i',j'}}^u(\phi(x, y)) = [\frac{M-1}{M}, 1] \times \{\frac{y}{2}\} .$$

For the stable manifold we have $\phi(W_{R_{i,j}}^s(x, y)) = \{2x\} \times [\frac{j-1}{2N}, \frac{j}{2N}]$, so

$$\begin{aligned} j \text{ even} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x\} \times [\frac{j-2}{2N}, \frac{j}{2N}] , \\ j \text{ odd} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x\} \times [\frac{j-1}{2N}, \frac{j}{2N}] , \end{aligned}$$

since $j' = \frac{j}{2}$ when j is even and $j' = \frac{j+1}{2}$ otherwise.

Case 2b: $\frac{1}{2} \leq x$

Since $x \geq \frac{1}{2}$, $(x', y') = \phi(x, y) = (2x-1, \frac{y+1}{2})$. From the fact that $\frac{1}{2} \leq x < \frac{M+1}{2M}$ it follows that $0 \leq x' < \frac{1}{M}$, so $i' = 1$ and

$$W_{R_{i',j'}}^u(\phi(x, y)) = [0, \frac{1}{M}] \times \{\frac{y+1}{2}\} .$$

For the stable manifold we have $\phi(W_{R_{i,j}}^s(x, y)) = \{2x-1\} \times [\frac{j-1}{2N} + \frac{1}{2}, \frac{j}{2N} + \frac{1}{2}]$. From the fact that $\frac{i-1}{N} < y < \frac{j}{N}$ it follows that $\frac{i-1}{2N} + \frac{1}{2} < y' < \frac{j}{2N} + \frac{1}{2}$. Thus, $j' = \frac{j+N}{2}$ if $j+N$ is even, and $j' = \frac{j+N+1}{2}$ if $j+N$ is odd. This gives us

$$\begin{aligned}
j + N \text{ even} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x - 1\} \times [\frac{j-2}{2N} + \frac{1}{2}, \frac{j}{2N} + \frac{1}{2}] , \\
j + N \text{ odd} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x - 1\} \times [\frac{j-1}{2N} + \frac{1}{2}, \frac{j+1}{2N} + \frac{1}{2}] .
\end{aligned}$$

Case 3: $\frac{1}{2} < \frac{i-1}{M}$

Since $x > \frac{i}{M} > \frac{1}{2}$, $(x', y') = \phi(x, y) = (2x - 1, \frac{y+1}{2})$ and so

$$\begin{aligned}
\phi(W_{R_{i,j}}^u(x, y)) &= [\frac{2i-2}{M} - 1, \frac{2i}{M} - 1] \times \{\frac{y+1}{2}\} \\
\phi(W_{R_{i,j}}^s(x, y)) &= \{2x - 1\} \times [\frac{j-1}{2N} + \frac{1}{2}, \frac{j}{2N} + \frac{1}{2}]
\end{aligned}$$

and

$$\begin{aligned}
W_{R_{i',j'}}^u(\phi(x, y)) &= [\frac{i'-1}{M}, \frac{i'}{M}] \times \{\frac{y+1}{2}\} \\
W_{R_{i',j'}}^s(\phi(x, y)) &= \{2x - 1\} \times [\frac{j'-1}{N}, \frac{j'}{N}] .
\end{aligned}$$

Note that if $\frac{i-1}{M} < x < \frac{i}{M} - \frac{1}{2M}$ then $\frac{2i-2}{M} - 1 < x' < \frac{2i-1}{M} - 1$, so $i' = 2i - 1 - M$. Similarly, if $\frac{i}{M} - \frac{1}{2M} \leq x < \frac{i}{M}$ then $i' = 2i - M$. Note also that $j' = \frac{j+N}{2}$ if $j + N$ is even and $j' = \frac{j+N+1}{2}$ when $j + N$ is odd. Thus,

$$\begin{aligned}
x < \frac{i}{M} - \frac{1}{2M} & : W_{R_{i',j'}}^u(\phi(x, y)) = [\frac{2i-2}{M} - 1, \frac{2i-1}{M} - 1] \times \{\frac{y+1}{2}\} , \\
\frac{i}{M} - \frac{1}{2M} \leq x & : W_{R_{i',j'}}^u(\phi(x, y)) = [\frac{2i-1}{M} - 1, \frac{2i}{M} - 1] \times \{\frac{y+1}{2}\} , \\
j + N \text{ even} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x - 1\} \times [\frac{j-2}{2N} + \frac{1}{2}, \frac{j}{2N} + \frac{1}{2}] , \\
j + N \text{ odd} & : W_{R_{i',j'}}^s(\phi(x, y)) = \{2x - 1\} \times [\frac{j-1}{2N} + \frac{1}{2}, \frac{j+1}{2N} + \frac{1}{2}] .
\end{aligned}$$

This completes the proof. \square

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